Ergodic Theory and Measured Group Theory
Lecture 1

The two subjects.

- **Ergodic theory** studies transformations of a measure space (typically, a standard probability space). Equivalently, it studies actions of (semi) groups on these spaces. The focus is the actions, their properties, whether or not two such actions are isomorphic.
- **Measured group theory** studies groups by studying its actions on measure spaces (again, typically standard prob. spaces).

"Groups, like people, are known by their actions."

— Some mathem.

This is the slogan of this subject. Here we try to understand how much of the group is "remembered" by its action on a measure space.

Focus of this course.

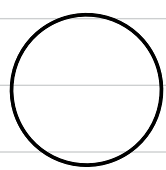
Group = cthly-infinite (discrete) group
Measure = probability measure
Functions on a measure space ⟹ measurable.
Null sets are ignored.

Standard probability spaces. We will denote a measure space by (X, μ), and we will write $(X, \mathcal{B}, \mu)$ if we want to indicate the σ-algebra.

Examples.

- $([0,1), \lambda)$, where λ is the Lebesgue measure.
- $([0,1)^n, \lambda_n)$, where $\lambda_n := \lambda^n$.
- $(2^{\mathbb{N}}, \nu^{\mathbb{N}})$, where $2 := \{0,1\}$, $2^{\mathbb{N}} :=$ the space of binary sequences, ν is a prob. measure on 2, e.g. $\nu(0) := \frac{1}{3}$ and $\nu(1) := \frac{2}{3}$. We will call this a coin-flip or Bernoulli measure. The fair coin-flip measure is $\nu(0) := \nu(1) := \frac{1}{2}$.
  The basic open sets of $2^{\mathbb{N}}$ (in the product topology) are of the form $[01101] := \{x \in 2^{\mathbb{N}} : x = 01101{*}{*}{*}\dots\}$, and $\nu^{\mathbb{N}}([01101]) := \nu(0)^2 \cdot \nu(1)^3$.

- $(\mathbb{N}^{\mathbb{N}}, \nu^{\mathbb{N}})$, where ν is a prob. meas. on $\mathbb{N}$.
- $([0,1)^{\mathbb{N}}, \lambda^{\mathbb{N}})$
- $S^1 :=$ the unit circle in $\mathbb{R}^2$, with the Lebesgue measure λ on S^1 where we view $S^1 \cong [0,1)$ by the exponential map.



Def. A Polish space is a topological space that is $\overbrace{\text{second ctbl}}^{\text{separable}\ \Updownarrow}$ (i.e. admits a ctbl open basis) and is completely metrizable (i.e. admits a complete metric producing the same topology).

Examples.

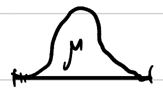
- $\mathbb{R}$, closed subsets of $\mathbb{R}$
- $[0,1)$, why? Because  $[0,1) \overset{\text{homeo}}{\cong} \overset{\text{complete}}{[0,\infty)}$
- $\mathbb{N}^{\mathbb{N}}$, why? $\forall x,y \in \mathbb{N}^{\mathbb{N}}$,
  $d(x,y) := 2^{-\Delta(x,y)}$, where $\Delta(x,y) :=$ the first index $i \in \mathbb{N}$ s.t. $x(i) \neq y(i)$ if $x \neq y$. O.w. $d(x,x) := 0$.
  This makes $\mathbb{N}^{\mathbb{N}}$ a complete metric space.
  The standard open basis is formed by the sets $[s) := \{x \in \mathbb{N}^{\mathbb{N}} : x \text{ begins with } s\}$, where s is a finite sequence of natural numbers.

Def. A measurable space $(X, \mathcal{A})$ is called standard Borel if $\mathcal{A} :=$ Borel σ-algebra of some Polish topology on X.

Def. A probability space $(X, \mathcal{A}, \mu)$ is called standard if $(X, \mathcal{A})$ is standard Borel.

Isomorphism theorem. (1) Any two unctbl standard Borel spaces are Borel isomorphic (e.g. $2^{\mathbb{N}}$, $\mathbb{R}$, $[0,1)$).

(2) Any two nonatomic standard prob. spaces are measure-isomorphic (e.g. to $([0,1), \lambda)$).

Example. Take a different prob. meas. on $[0,1)$, say 

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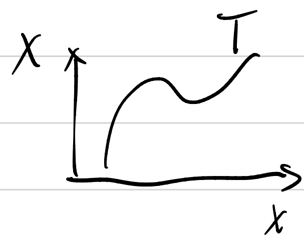
Then $\exists$ Borel bijection $f : [0,1) \to [0,1)$ a.e. s.t. $f_* \mu = \lambda$, where $f_* \mu$ is the push-forward measure of μ by f.

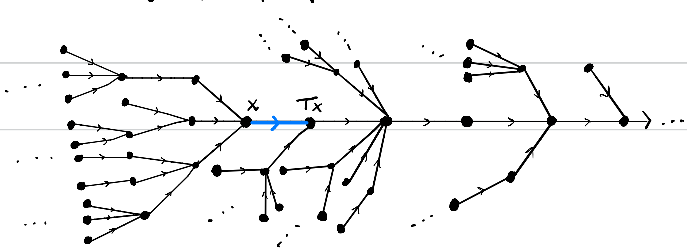
Probability measure preserving (pmp) transformations.

Let (X, μ) be a prob. space and let $T : X \to X$ be a measurable transformation.

How an analyst thinks of T:

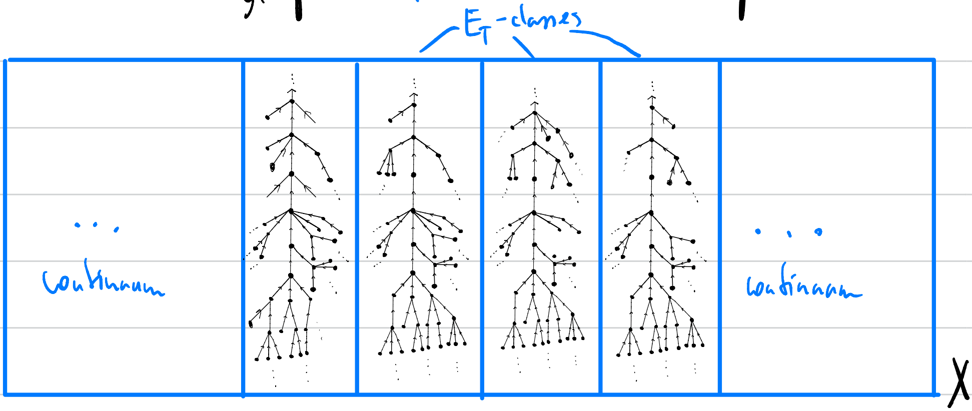


How a DST-ist thinks of T: also a graph, but as in combinatorics



Here, $x \in X$ and drawn is the connected component of x. We call it an orbit of T.

The whole graph G_T on the vertex space X.



The equiv. rel. of being in the same connected component is called the orbit equiv. rel. of T and denoted by E_T. More precisely, $\forall\, x, y \in X$,

$$x \, E_T \, y \;:\Leftrightarrow\; \exists\, n, m \in \mathbb{N},\; T^n(x) = T^m(y).$$

Obs. Any measurable $T : (X, \mu) \to (X, \mu)$ is Borel on a conull subset of X.

Proof. The Borel σ-algebra of X is ctbl'y-generated, say by a ctbl family $\mathcal{U}$ of subsets of X, and $\forall\, B \in \mathcal{U}$, $T^{-1}(B)$ is Borel modulo a null set, so throwing out these ctbly-many sets from X makes T a Borel map. $\square$

Thus, WLOG, we assume T is Borel, i.e. $T^{-1}(\text{Borel})$ is Borel.

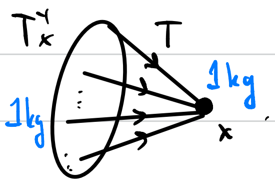
Note that if T is Borel, then G_T and E_T are Borel subsets of X^2:

$$x \, G_T \, y \;:\Longleftrightarrow\; T(x) = y$$
$$x \, E_T \, y \;:\Longleftrightarrow\; \exists n, m \in \mathbb{N},\ T^n(x) = T^m(y),$$

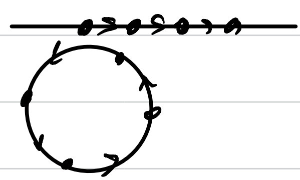
both are Borel definitions because of a theorem in descriptive set theory that says that a function is Borel $\Leftrightarrow$ its graph is a Borel set.

Def. A meas. transformation $T : (X, \mu) \to (X, \mu)$ is said to be *measure-preserving* if the probability of a random point x being in a given Borel set B is $=$ to the prob. of $Tx \in B$. In other words, $\mu(B) = \mu(T^{-1}B)$.

The way to think about this is that the "weight" of each point $x \in X$ is equal to the total weight of the set $T^{-1}x$



if T is one-to-one, then each connected component of X is either



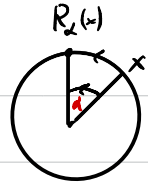
or

and the weights of all points are equal.

Examples.

○ **Rotation** $R_\alpha : S^1 \to S^1$, where $\alpha \in [-\pi, \pi)$.

$$e^{\beta i} \mapsto e^{(\beta+\alpha)i}$$



If α/π is rational, then R_α is periodic, i.e. $\exists n$ s.t. $R_\alpha^n = id_{S^1}$, so the orbits are just finite cycles.

If α/π is irrational, then R_α is aperiodic and the orbits are $\mathbb{Z}$-lines 

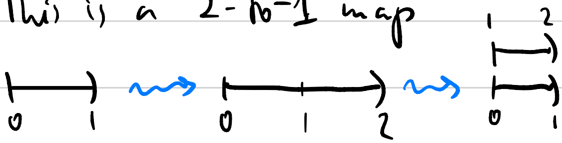
Each orbit is dense in S^1.

R_α is pmp because it doesn't change the length of arcs.

○ **The baker's map** $b_2 : [0,1) \to [0,1)$

$x \mapsto 2x \mod 1$ = erase the first digit in the binary rep. of x.

This is a 2-to-1 map



This is pmp because for any interval $I \subseteq [0,1)$, $b_2^{-1}(I)$ is two intervals each half of the length of I:

$$|I_1| = |I_2| = \tfrac{1}{2}|I|.$$